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Percolation theory based connectivity and latency analysis of cognitive radio ad hoc networks

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Abstract This paper investigates the dynamic connectivity and transmission latency of cognitive radio ad-hoc networks (secondary networks) coexisting with licensed networks (primary networks) that experience time-varying on-off links. It is shown that there exists a critical density λ_s^* such that if the density of secondary networks is larger than λ_s^*, the secondary network percolates at all time $t > 0$, i.e., there exists always an infinite connected component in the secondary network under the time-varying spectrum availability. Furthermore, the upper and lower bounds of λ_s^* are derived and it is shown that they do not depend on the random locations of primary and secondary users, but only on the network parameters, such as active/inactive probability of primary users, transmission range, and the user density. Moreover, due to the dynamic behavior of the unoccupied spectrum, the secondary network can be disconnected at all times. It is proven that it is still possible for a SU to transfer its message to any destination with a certain delay with probability one. This delay is shown to be asymptotically linear in the Euclidean distance between the transmitter and receiver.

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1 Introduction

Cognitive radio ad-hoc networks (CRAHNs) enable the unlicensed users (secondary users) to utilize the spectrum holes unoccupied by the licensed users (primary users) so that the limited spectrum resource is significantly conserved [1]. Recent research efforts have been focused on designing effective spectrum management mechanisms, e.g., spectrum sensing, spectrum decision, spectrum sharing, and spectrum mobility. However, to make CRAHNs function properly and meet performance requirements, CRAHNs have to maintain connectivity under the dynamically changing spectrum holes.

Conventionally, the full connectivity is examined for the wireless networks either made of a finite number of nodes [2, 15] or deployed in a finite area [6]. In this case, the full connectivity ensures that each pair of nodes in the network is connected by at least one path. However, in a large-scale CRAHN deployed in a sufficiently large area, this full connectivity criterion may be overly restrictive or difficult to achieve because of the complicated radio environment, unplanned network topology, and severe impacts from coexisting networks. In this paper, we investigate the connectivity of large-scale CRAHNs from a different perspective. A CRAHN (secondary network) is considered to be functional and connected if it contains an extremely large connected component such that each secondary user (SU) in this component can connect to an extremely large number of SUs. Without such a component, the secondary network would be partitioned into small fragments, and thus becomes unconnected and unusable. From this perspective, the key issue of the network connectivity is to characterize the conditions under which there exists an extremely large connected component. A powerful technique for solving this issue comes from the percolation theory [11]. Recently, the percolation theory has been proven to be a very useful tool for the analysis of large-scale wireless networks [4, 9].

Percolation theory, especially the continuum percolation theory [11], is targeted at the random geographic graph in which the nodes are randomly distributed with a certain density λ, and two nodes are connected if their mutual distance is less than a threshold. The key result of continuum percolation theory concerns a phase transition phenomenon where the network exhibits fundamentally different behavior for the density λ below and above some critical density λ_c. If $\lambda > \lambda_c$, the network is in the supercritical phase, and a connected component comprising an infinite number of nodes exists with probability one. If $\lambda < \lambda_c$, the network is in the subcritical phase such that there only exists connected components containing a finite number of nodes.

In this paper, we use dynamic percolation processes to study dynamic connectivity in the secondary network under the time and location varying spectrum. It is shown that the phase transition can still occur in the secondary network under the dynamically changing radio environment and the interference constraints. Specifically, it is determined that there exists a critical density λ_s^* such that if the density of secondary networks is larger than λ_s^*, the secondary network achieves percolation-based connectivity at all times, i.e., as time proceeds there always exists an infinite connected component in the secondary network with probability one. Furthermore, the upper and lower bounds of λ_s^* are derived and it is shown that they do not depend on the random locations of primary and secondary users, but only on the network parameters, such as active/inactive probability.

Moreover, due to the time and location varying spectrum availability, a certain delay is incurred for the transmission in the secondary network. Whether this delay has rigorous bounds directly affects the QoS satisfaction levels of secondary users. This problem is investigated by using subadditive ergodic theorem, and it is proven that this delay is asymptotically linear in the Euclidean distance between the sender and receiver. This implies that even if there is no connectivity of all SUs at all times, it is still possible for a SU to transfer its data to any destination in a bounded time with probability one.

So far, percolation-based connectivity has been investigated in large-scale distributed networks such as ad-hoc and wireless sensor networks [3, 4, 9]. Recently, the connectivity of cognitive radio ad-hoc networks was studied [13]. This work, however, does not take into account the time-varying spectrum availability induced by the primary users, a key characteristic that distinguishes cognitive radio networks from the general networking paradigms. In [14], the transmission delay of cognitive radio users is investigated under the assumption that the time of primary networks is slotted and the active primary users are i.i.d. across slots. In contrary, in this paper, we adopt a more general model for primary network. In this model, a primary user simply alternates between busy and idle states and the length of busy and idle states can follow arbitrary distributions. Thus, the assumption of slotted time and i.i.d. active primary users across slots is not necessary. This complicates the connectivity and delay analysis.

The rest of this paper is organized as follows. In Sect. 2, we introduce formally the network models. In Sect. 3, we prove the existence of the critical density for percolation based connectivity in the secondary network. Then, the upper and lower bounds of the critical density are derived. In Sect. 4, we study the transmission latency in the secondary network. Finally, Sect. 5 concludes this paper.

2 Network models

2.1 Random disk graph

We use random disk graph of Poisson point process to model large scale networks. In this graph, the nodes are distributed according to homogeneous Poisson point process with a certain density λ. Let $X_{i=1}^n$ denote the random locations of the nodes $\{1, 2, ..., n\}$. With each node i as the center, place a disk with radius $a/2$. If the two disks of i and j touch, i.e., if $\|X_i - X_j\| \le a$, then an edge exists between i and j. These edges produce a random disk graph of density λ with range a, denoted by $G(\lambda, a)$.

As discussed above, there is a critical density λ_c such that if $\lambda > \lambda_c$, there exists an infinite connected component (a connected component with an infinite number of nodes) in $G(\lambda, a)$. The exact value for λ_c is not known. For the random disk graph with $a = 1$, i.e., $G(\lambda, a)$, simulation results show that $1.43 < \lambda_c < 1.44$ [12].

2.2 Primary network model

We model the PU senders by a random disk graph of density λ_p with range $2R$, denoted by $G(\lambda_p, 2R)$. R is the transmission radius of each primary user. Note that the range of $G(\lambda_p, 2R)$ is chosen to be $2R$ so that $G(\lambda_p, 2R)$ can model the continuous interference region generated by the PU senders. Let each PU sender associated with a PU receiver, which is uniformly distributed within the transmission radius of the sender. From [7], the PU receivers also follow homogeneous Poisson point process with density λ_p, except that the two Poisson point processes are dependent of each other.

Let each PU sender associated with an independent and identically distributed (i.i.d.) alternating renewal process, denoted by $S_p(t)$, which alternates between two states: the ON state, during which the PU is active; and the OFF state, during which the PU is inactive. A PU receiver is active/inactive if its associated PU sender is active/inactive. Denote the length of ON and OFF state by $\tau_0 > 0$ and $\tau_1 > 0$ respectively. Assume $\tau_0(\tau_1)$ follows an arbitrary distribution with finite expectation, i.e., $E(\tau_0) < \infty$ and $E(\tau_1) < \infty$. Therefore, the marginal distribution of $S_p(t)$ is

$$\begin{aligned}\Pi_0 &= \Pr(S_p(t)=0) = \frac{E(\tau_0)}{E(\tau_0)+E(\tau_1)}\\ \Pi_1 &= \Pr(S_p(t)=1) = \frac{E(\tau_1)}{E(\tau_0)+E(\tau_1)}\end{aligned} \tag{1}$$

2.3 Secondary network model

We model the secondary network by a random disk graph of density λ_s with range r, denoted $G(\lambda_s, r)$. r is the transmission radius of each secondary user. Let $\{X_{i=1}^n\}$ denote the random locations of the $\{\mathrm{SU}_{i=1}^n\}$. Since the Euclidean distance is the connection criterion in $G(\lambda_s, r)$, $G(\lambda_s, r)$ actually models a standalone secondary network without considering the impact of primary networks. This means $G(\lambda_s, r)$ only contains geographic links. A geographic link exists between SU_i and SU_j if the Euclidean distance between them is less than r, i.e., if $\|X_i - X_j\| < r$.

2.4 Connection criterion under dynamic spectrum activity

Definition 1 We say SU_i and SU_j are dynamically connected if there exists an undirected functional link between i and j. A functional link exists if the following conditions are fulfilled

1) $\|X_i - X_j\| < r$
2) Both SU_i and SU_j are outside the transmission range R of every *active* PU sender.
3) There is no *active* PU receiver residing in the transmission range r of SU_i and SU_j.

The first condition ensures that there is a geographic link between the SUs. The second condition guarantees that the active PUs do not generate any interference to SU_i and SU_j so that the two SUs can identify an available channel to communicate with each other. The third condition prohibits the communications between the two SUs interfering the active PU receivers.

3 Dynamic connectivity of cognitive radio ad-hoc networks

In this section, we study the connectivity in a secondary network $G(\lambda_s, r)$ that coexists with a primary network $G(\lambda_p, 2R)$. Since the primary users are distributed randomly and switch dynamically between the active and inactive states, the link availability between SUs is also changing over time and location. To capture this characteristic, we use dynamic percolation processes to study the connectivity in the secondary network. Denote the critical density of $G(\lambda_s, r)$ and $G(\lambda_p, 2R)$ by λ_s^c and λ_p^c, respectively. Denote the sampled primary network at time t by $G(\lambda_p, 2R, S_p(t))$. Thus, $G(\lambda_p, 2R, S_p(t))$ consists of the all the active primary users in $G(\lambda_p, 2R)$ with their associated links. Denote the sampled secondary network at time t by $G(\lambda_s, r, S_p(t))$. That is, $G(\lambda_s, r, S_p(t))$ only comprises the secondary users in $G(\lambda_s, r)$ that have functional links. Then, we have the following theorem regarding the secondary network connectivity.

Theorem 1 *Given a secondary network $G(\lambda_s, r, S_p(t))$ coexisting with a primary network $G(\lambda_p, 2R, S_p(t))$, there exists a critical density $\lambda_s^c < \lambda_s^* < \lambda_s'(\Pi_1\lambda_p^c, r, R)$ such that if $\lambda_s > \lambda_s^*$ and $\lambda_p < \Pi_1\lambda_p^c$, then with probability one, there exists an infinite connected component in the secondary network $G(\lambda_s, r, S_p(t))$ for all times $t > 0$.*

Note that the upper bound of $\lambda_s^*, \lambda_s(\Pi_1\lambda_p^c, r, R)$, is a function in terms of the network parameters, which include the active probability Π_1 of primary users, the critical density λ_p^c of primary network $G(\lambda_p, 2R)$, and the transmission radius R and r. This means that the critical density is only related to the network settings and independent from the random locations of primary and secondary users.

To prove Theorem 1, we first investigate static continuum percolation on the secondary network. Then, we take the dynamic behavior into account. The similar strategy is also used to study the performance of the energy management mechanisms in wireless sensor networks [9]. Consider a primary network $G(\lambda_p, 2R)$. Assume each PU is active independently with probability Π_1. According to thinning theory of Poisson point process [7], the primary network can be represented by a random disk graph $G(\Pi_1\lambda_p, 2R)$ with density $\Pi_1\lambda_p$. Denote $G(\lambda_s, r, \Pi_1)$ the secondary network coexisting with the primary network $G(\Pi_1\lambda_p, 2R)$.

Proposition 1 *Given a secondary network $G(\lambda_s, r, \Pi_1)$, there exists a value $\lambda_s^c < \lambda_s^* < \lambda_s'(\Pi_1\lambda_p^c, r, R)$ such that if $\lambda_s > \lambda_s^*$ and $\lambda_p < \Pi_1\lambda_p^c$, then $G(\lambda_s, r, \Pi_1)$ is percolated.*

To prove Proposition 1, we employ a mapping between continuum percolation on the continuous plane $\mathbb{R}^2$ and bond percolation on a lattice. The mapping is as follows. We begin by placing a lattice L with the edge length d on the plane $\mathbb{R}^2$. All the vertices of L are located at ($d \times i$, $d \times j$), where $(i,j) \in \mathbb{Z}$. We choose $d = r/\sqrt{5}$ so that the maximum distance between any two SUs in adjacent squares is not greater than the transmission range r. That is, there exists a geographic link between the two SUs. For each vertical edge e in L, let its two end vertices be $(e_x \times d, e_y \times d)$ and $(e_x \times d, e_y \times d + d)$.

Definition 2 A vertical edge e of L is said to be open if the following conditions are satisfied:

1) both squares adjacent to e contains at least one SU;
2) the rectangle $R_e = [e_x d - d - \lceil R/d \rceil d, e_x d + d + \lceil R/d \rceil d] \times [e_x d - \lceil R/d \rceil d, e_x d + d + \lceil R/d \rceil d]$ does not contain any active PU senders;
3) the rectangle $[R_e{}' = [e_x d - d - \lceil r/d \rceil d, e_x d + d + \lceil r/d \rceil d] \times [e_x d - \lceil r/d \rceil d, e_x d + d + \lceil r/d \rceil d]$ does not contain any active PU receivers.

Define similarly the open horizontal edge of L by rotating the rectangle R_e and $R_e{}'$ by 90 degree, respectively. Next, we construct the dual lattice of L, which is denoted by L'. L' is obtained from L in the following way. A vertex is placed in the center of each square of L, and two such neighboring vertices are joined by a straight line segment. This line segment becomes an edge of L'. As L is a square lattice, the dual lattice L' is the same lattice shifted by (d/2, d/2). Note that there is a one-to-one correspondence between the edges in L and edges in L', since each edge of L is crossed by a unique edge of L'.

Definition 3 An edge of L' is said to be open if and only if its corresponding edge of L is open.

Definition 4 A path (in L or L') is said to be open if and only if all its edges are open; a path (in L or L') is said to be close if and only if all its edges are close.

The basic idea of the proof for Proposition 1 is to translate the presence of continuum percolation on $G(\lambda_s, r, \Pi_1)$ into the presence of bond percolation on the lattice L'. More specifically, we first show that the secondary network $G(\lambda_s, r, \Pi_1)$ will have an infinite connected component on the continuous plane $\mathbb{R}^2$ if bond percolation occurs on L', i.e., if there exists an infinite open path on L'. Then, we prove that under certain conditions, the bond percolation indeed occurs on L'. Before giving the proof of Proposition 1, we need the following lemmas.

Lemma 1 *If there exists an open edge in L', then $G(\lambda_s, r, \Pi_1)$ contains at least two mutually connected SUs, which reside in the region covered by two adjacent squares along the open edge.*

Proof We consider an open edge e' of L'. By our construction of L', each vertex of e' is located at the center of a square of L. Therefore, along edge e', we find two adjacent squares that satisfy the conditions given in Definition 2. Let M denote the region covered by the two adjacent squares. Then, the first condition ensures that M contains at least two SUs denoted by SU_1 and SU_2 that can reach each other within the transmission range r. The second condition guarantees that the SUs in M have available spectrum to communicate since R_e is the minimum region outside which the presence of PU senders cannot generate interference in the region M. The third condition prohibits SU_1 and SU_2 interfering PU receivers since $R_e{}'$ is the maximum region in which the communications between SU_1 and SU_2 can induce interference for PU receivers. According to the connectivity conditions described in Definition 1, there exists at least two SUs that can connect to each other in a certain region M. □

Lemma 2 *If there exists an infinite open path in L', there exists an infinite connected component in $G(\lambda_s, r, \Pi_1)$.*

Proof We consider an infinite open path $P_\infty = \{e_i'\}_{i=1}^{\infty}$ in L'. Along each edge e_i', there exists two adjacent squares in the dual lattice L. Therefore, along P_∞, there exists a sequence of squares $\{S_i\}_{i=1}^{\infty}$ in L such that any two consecutive squares, denoted by S_i and S_{i+1}, are adjacent. By Lemma 1, the region comprising S_i and S_{i+1} contains at least two mutually connected SUs that belong to $G(\lambda_s, r, \Pi_1)$ Thus, the sequence of squares $\{S_i\}_{i=1}^{\infty}$ forms an infinite connected component in $G(\lambda_s, r, \Pi_1)$. □

Lemma 3 *If $\lambda_p > \Pi_1 \lambda_p^c$, with probability one there exists no infinite connected component in $G(\lambda_s, r, \Pi_1)$.*

Proof To prove there exists no infinite connected component in $G(\lambda_s, r, \Pi_1)$, it is sufficient to prove that any connected component in $G(\lambda_s, r, \Pi_1)$ only contains a limited number of SUs. More specifically, denote by W the connected component containing the origin, we will prove that $|W| < \infty$ with probability one under the condition $\lambda_p > \Pi_1 \lambda_p^c$. Note that in a random disk graph induced by a homogeneous Poisson point process, all the nodes are probabilistically indistinguishable, and thus an arbitrary node can be selected as the origin.

The basic idea of the proof is to show that if condition $\lambda_p > \Pi_1 \lambda_p^c$ is satisfied, the connected component W in secondary network $G(\lambda_s, r, \Pi_1)$ is certainly surrounded by some continuous interference region of the primary network $G(\Pi_1 \lambda_p, R)$ such that all paths starting from the connected component W are blocked by the interference region and thus completely constrained inside a finite area.

We start by placing a new square lattice L_p on $\mathbb{R}^2$, with the edge length d_p. Consider a sequence $\{G_i\}_{i\geq 1}$ of annuli around the origin. Each annulus G_i is made up of four rectangles

$$\begin{aligned} A_i^+ &= [-d_p 2^i, d_p 2^i] \times [d_p 2^{i-1}, d_p 2^i] \\ A_i^- &= [-d_p 2^i, d_p 2^i] \times [-d_p 2^i, d_p 2^{i-1}] \\ B_i^+ &= [d_p 2^{i-1}, d_p 2^i] \times [-d_p 2^i, d_p 2^i] \\ B_i^- &= [-d_p 2^i, -d_p 2^{i-1}] \times [-d_p 2^i, d_p 2^i] \end{aligned} \tag{2}$$

We say that $A_i^+(A_i^-)$ is closed if $A_i^+(A_i^-)$ is crossed from left to right by a connected component in $G(\Pi_1\lambda_p, 2R)$. Similarly, we declare that $B_i^+(B_i^-)$ is closed if $B_i^+(B_i^-)$ is crossed from bottom to top by a connected component in $G(\Pi_1\lambda_p, 2R)$. A example of closed rectangle is given in Fig. 1. The structure of the annulus is shown in Fig. 2.

Let $\tilde{A}_i^+, \tilde{A}_i^-, \tilde{B}_i^+$, and $\tilde{B}_i^-$ be the events that A_i^+, A_i^-, B_i^+, and B_i^- are closed, respectively. According to Corollary 4 in [11], when $\lambda_p > \Pi_1\lambda_p^c$, i.e., $G(\Pi_1\lambda_p, R)$ is in the supercritic phase, we can choose d_p large enough so that events $\tilde{A}_i^+, \tilde{A}_i^-, \tilde{B}_i^+$, and $\tilde{B}_i^-$ occur with the probability arbitrarily close to 1. This means that for any $0 < \delta < 1$, there always exists d_p' so that if $d_p \geq d_p'$, $\Pr(\tilde{A}_i^+) = \Pr(\tilde{A}_i^-) = \Pr(\tilde{B}_i^+) = \Pr(\tilde{B}_i^-) \geq \delta$.

If events $\tilde{A}_i^+, \tilde{A}_i^-, \tilde{B}_i^+$, and $\tilde{B}_i^-$ occur simultaneously, the annulus G_i must contain a continuous interference region generated by the active PU senders, and hence the connected component in $G(\lambda_s, r, \Pi_1)$ is necessarily surrounded by the outer boundary of G_i. Denote the latter event by $\tilde{G}_i$. Since $\tilde{A}_i^+, \tilde{A}_i^-, \tilde{B}_i^+$, and $\tilde{B}_i^-$ are dependent but increasing events (For the definition of increasing events, please refer to Chapter 2.3 [11]), utilizing Fortuin–Kasteleyn–Ginibre (FKG) inequality [11] yields

$$\begin{aligned} \Pr(\tilde{G}_i) &= \Pr(\tilde{A}_i^+ \cap \tilde{A}_i^- \cap \tilde{B}_i^+ \cap \tilde{B}_i^-) \\ &\geq \Pr(\tilde{A}_i^+)\Pr(\tilde{A}_i^-)\Pr(\tilde{B}_i^+)\Pr(\tilde{B}_i^-) \geq \delta^4 \end{aligned} \tag{3}$$

Thus, we have $\sum_{i=1}^{\infty}\Pr(\tilde{G}_i) \geq \sum_{i=1}^{\infty}\delta^4 = \infty$. However, our construction of the annuli $\{G_i\}_{i\geq 1}$ guarantees that events $\{\tilde{G}_i\}_{i\geq 1}$ are independent. Therefore, by the Borel-Cantelli lemma [11], there exists $j < \infty$ so that $\tilde{G}_j$ occurs with the probability 1. This means that there must exist a $G_{j<\infty}$ such that its outer boundary $[-d_p 2^j, d_p 2^j] \times [-d_p 2^j, d_p 2^j]$ surrounds the connected component W. Since $|W|$ can not be greater than the total number of SUs within G_js outer boundary, we have $E(|W|) \leq 16\Pi_1\lambda_p d_p^2 j^2 < \infty$. This implies $P(|W| < \infty) = 1$. □

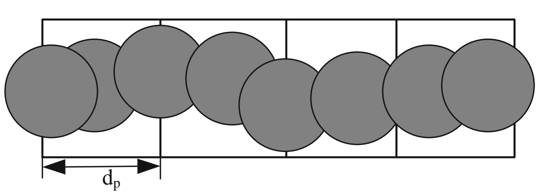

Fig. 1 Closed rectangle

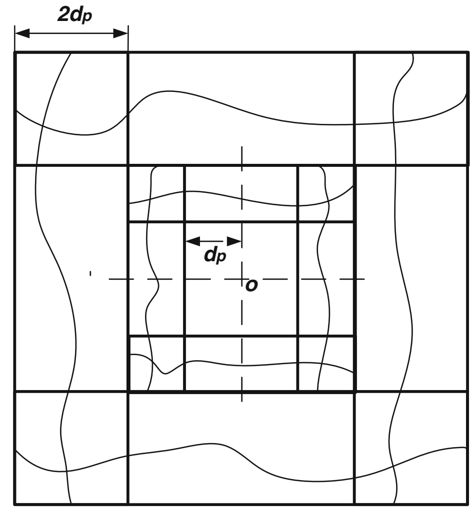

Fig. 2 The annulus G_1 (*inside*) and G_2 (*outside*). Each annulus has four closed (*crossed*) rectangles

Using Lemmas 1–3, we give the proof for Proposition 1.

Proof (of Proposition 1) Let E_1, E_2, and E_3 be the events when the conditions (i), (ii), and (iii) in Definition 2 are satisfied, respectively. Let C_e denote the event that an edge e is closed. The probability that C_e occurs is upper bounded by

$$\begin{aligned} \Pr(C_e) &= 1 - \Pr(E_1 \cap E_2 \cap E_3) \\ &\overset{a}{=} 1 - \Pr(E_1)\Pr(E_2 \cap E_3) \\ &\overset{b}{\leq} 1 - \Pr(E_1)\Pr(E_2)\Pr(E_3) \\ &= 1 - (1 - e^{-d^2\lambda_s})^2 e^{-|R_e|\Pi_1\lambda_p} e^{-|R_e'|\Pi_1\lambda_p} \end{aligned} \tag{4}$$

where $|R_e| = (2 + 2\lceil R/d\rceil)(1 + 2\lceil R/d\rceil)d^2$ is the area of R_e, and $|R_e'| = (2 + 2\lceil r/d\rceil)(1 + 2\lceil r/d\rceil)d^2$ is the area of R_e'. The equality a in (4) comes from the independence of the locations of primary and secondary users. The inequality b is due to the fact that the Poisson point process of PU senders is correlated with that of PU receivers such that E_2 and E_3 are dependent. However, both E_2 and E_3 are decreasing events. By FKG inequality [11], we obtain $\Pr(E_2E_3) \geq \Pr(E_2)\Pr(E_3)$.

As seen in (4), the open/closed state of a particular edge e depends on the Poisson point processes in some regions (R_e and R_e'). Thus, the states of the edges in L are not independent from each other. More specifically, let $R_e^{\max}$ be the region satisfying the condition

$$R_e^{\max} = \arg \max_{x \in \{R_e, R'_e\}} |x| \tag{5}$$

The states of any two edges e_i and e_j are dependent if and only if $R_{e_i}^{\max}$ and $R_{e_j}^{\max}$ are overlapped. Thus, all the edges correlated with e necessarily reside in the correlation region depicted in Fig. 3. Consequentially, in the lattice L, the total number Λ of the edges correlated with e can be computed by

$$\begin{aligned}\Lambda = &(4Ld + 2d + 1) \times (4Ld + d) \\ &+ (4Ld + d + 1) \times (4Ld + 2d) - 1,\end{aligned} \tag{6}$$

where $L = \lceil \max\{R, r\}/d \rceil$.

Now, let us consider a path $P_n = \{e_i\}_{i=1}^n$ of length n in L. By (6), each $e_i \in P_n$ has maximum correlated edges that belong to P_n. Therefore, there exist at least $m \geq n/\Lambda$ edges in P_n, e.g., $\{e_j\}_{j=1}^m \subseteq \{e_i\}_{i=1}^n$, such that their states are independent from each other. Let X_{e_i} denote the event that e_i is closed. Then, the probability that the path P_n is closed is upper bounded by

$$\Pr(\text{closed } P_n) = \Pr\left(\bigcap_{i=1}^n X_{e_i}\right) \leq \Pr\left(\bigcap_{i=1}^m X_{e_i}\right) \leq q^{\frac{n}{\Lambda}}, \tag{7}$$

where $q = \Pr(C_e)$ as given in (4).

By the duality between L and L', a key observation is that if an open path starting from a vertex (e.g., the origin) in L' is finite, the origin is necessarily surrounded by a closed circuit (a closed path with the same starting and ending vertex) in the dual lattice L. Hence, by letting the latter event be O_L, the probability that there exists an infinite open path starting from the origin is $1 - \Pr(O_L)$. Furthermore, from (7), we have

$$\Pr(O_L) = \sum_{n=2}^{\infty} \sigma(n) \Pr(\text{closed } P_{2n}) \leq \sum_{n=2}^{\infty} \sigma(n) q^{\frac{n}{\Lambda}}, \tag{8}$$

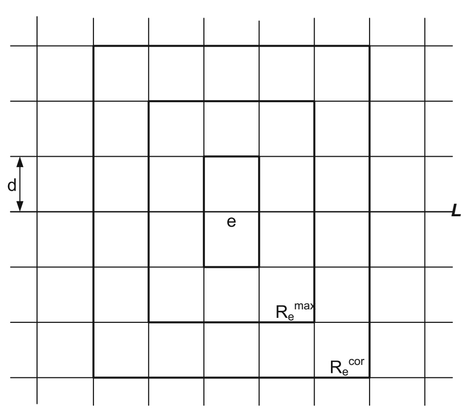

Fig. 3 Edge e and its correlation region R_e^{corr} assuming $R > r$ and $R = d$

where $\sigma(n)$ is the number of closed circuits of the length $2n$ surrounding the origin. It is easy to show that $\sigma(n)$ is upper bounded by

$$\sigma(n) \leq (n-1) 3^{2(n-1)}. \tag{9}$$

Hence, we have

$$\Pr(O_L) = \sum_{n=2}^{\infty} (n-1) 3^{2(n-1)} q^{\frac{2n}{\Lambda}} = \frac{9q^{4/\Lambda}}{(1 - 9q^{2/\Lambda})^2}. \tag{10}$$

Therefore, from (10) and (4), if $q = \Pr(C_e) < 1/(2\sqrt{3})$, i.e.,

$$\begin{aligned}&\lambda_s > \lambda_s(\Pi_1 \lambda_p, R, r) \\ &= \frac{5}{r^2} \ln \left[1 - \sqrt{\left(1 - (\sqrt{6}/3)^{\Lambda}\right) e^{(|R_e| + R'_e|)\Pi_1 \lambda_p}} \right]^{-1}\end{aligned} \tag{11}$$

then $\Pr(O_L)$ converges to a number less than one. As a consequence, the probability that there exists an infinite open path starting from the origin in L' is positive. According to Kolmogorovs zero-one law, this implies that an infinite path exists in L' with probability one. From Lemma 2, the existence of an infinite path in L' further implies the existence of an infinite connected component in $G(\lambda_s, r, \Pi_1)$. Therefore, $G(\lambda_s, r, \Pi_1)$ percolates.

In the above, we prove that if $\lambda_s > \lambda'_s(\Pi_1 \lambda_p, R, r)$, the secondary network $G(\lambda_s, r, \Pi_1)$ contains an infinite connected component. This function $\lambda'_s(\Pi_1 \lambda_p, R, r)$ is, thus, the upper bound on the critical density λ_s^*. However, Lemma 3 implies that if $G(\lambda_s, r, \Pi_1)$ percolates, λ_p is necessarily less than $\Pi_1 \lambda_p^c$. Furthermore, since $\lambda'_s(\Pi_1 \lambda_p, R, r)$ is an increasing function of λ_p, as indicated in (11). Therefore, λ_s^* is upper bounded by

$$\lambda_s^* \leq \lambda'_s(\Pi_i \lambda_p^c, R, r) \tag{12}$$

To obtain the lower bound on λ_s^*, we consider a standalone secondary network $G(\lambda_s, r)$. By coupling argument, all the connected components in $G(\lambda_s, r, \Pi_1)$ are also in $G(\lambda_s, r)$. It is known that if $\lambda_s < \lambda_s^c$, there exists no infinite connected components in $G(\lambda_s, r)$, and hence all the connected components in $G(\lambda_s, r, \Pi_1)$ are finite. This value λ_s^c is thus a lower bound on λ_s^*, i.e.,

$$\lambda_s^* \geq \lambda_s^c \tag{13}$$

□

Now, we are in the position to prove Theorem 1. We apply the similar method as the one used in the proof of dynamic bond percolation [8].

Proof (of Theorem 1) We first show that if $\lambda_s > \lambda'_s(\Pi_1 \lambda_p^c, R, r)$, then there exists an infinite connected component in $G(\lambda_s, r, S_p(t))$ for all $t > 0$ with probability one. Then, we show that if $\lambda_s < \lambda_s^c$, no infinite connected component exists in $G(\lambda_s, r, S_p(t))$ for all $t > 0$ with probability one.

Assume $\lambda_s > \lambda'_s(\Pi_1\lambda_p^c, R, r) = \lambda'_s((1-\Pi_0)\lambda_p^c, R, r)$. Since $\lambda'_s(x, R, r)$ is an increasing function of x, there exists $y > (1-\Pi_0)\lambda_p^c$ such that $\lambda_s = \lambda'_s(y, R, r) > \lambda'_s((1-\Pi_0)\lambda_p^c, R, r)$. Let $0 < \varepsilon < 1$ such that $y > (1-(1-\varepsilon)\Pi_0)\lambda_p^c$. Since the length of the OFF period is nonzero, we can find $\delta > 0$ such that $\Pr(S_p^\delta = 0 | S_p(t=0)) > 1-\delta$, where $S_p^\delta = \min_{t\in[0,\delta]}(S_p(t))$. Then, $\Pr(S_p^\delta = 0) > (1-\varepsilon)\Pi_0$. Therefore, we have

$$\lambda_s > \lambda'_s(\Pi_1\lambda_p, R, r) > \lambda'_s((1-\Pr(S_p^\delta = 0))\lambda_p, R, r) \quad (14)$$

This means that an infinite connected component exists in $G(\lambda_s, r, S_p(t))$ for all $t \in [0, \delta]$. Let E_i be the event that an infinite connected component exists for all $t \in [i\delta, (i+1)\delta]$, and E_i^c be its complement. Then, we have

$$\Pr\left(\bigcap_i E_i\right) = 1 - \Pr\left(\bigcup_i E_i^c\right) \geq 1 - \sum_i \Pr(E_i^c) = 1 \quad (15)$$

This indicates that there exists an infinite connected component in $G(\lambda_s, r, S_p(t))$ for all $t > 0$.

If $\lambda_s < \lambda_s^c$, then $G(\lambda_s, r)$ is in the subcritical phase. Since for any $t > 0, G(\lambda_s, r, S_p(t)) \subseteq G(\lambda_s, r)$, then $G(\lambda_s, r, S_p(t))$ is in the subcritical phase at all times, i.e., no infinite connected component exists in $G(\lambda_s, r, S_p(t))$ for all $t > 0$. □

4 Latency of CRAHNs

In the previous section, we have shown that there exists a critical density λ_s^* such that if $\lambda_s > \lambda_s^*$ and $\lambda_p < \Pi_1\lambda_p^c, G(\lambda_S, r, S_p(t))$ percolates with probability one. However, the density and the active probability of PUs increases, the opportunity for SUs to identify the available links dramatically decreases. Consequentially, maintaining the connectivity or percolation in the secondary network at all times becomes more and more difficult. More specifically, we have shown in Lemma 3 that if the primary network density λ_p exceeds a certain value $\Pi_1\lambda_p^c$, with probability one, all the paths between any pair of SUs will be blocked by some interference region generated by PUs, and thus $G(\lambda_s, r, S_p(t))$ is partitioned at each time $t > 0$. However, we will show that even if there is no connectivity of all SUs at all times, the message from a SU u could still be eventually relayed to any SU v with a bounded delay with probability one. More specifically, we will prove that even if $G(\lambda_s, r, S_p(t))$ never percolates, the message transmission latency only scales linearly with the Euclidean distance between two SUs, as long as the two SUs are in the infinite connected component of $G(\lambda_s, r)$, i.e., as long as there exists a path made of geographic links between the two SUs.

The main tool used in the following proofs is the subadditive ergodic theorem [10], which is originally used to study the first passage percolation processes [5]. Recently, the subadditive ergodic theorem is proven to be a powerful tool to analyze the latency in dynamic systems. For example, this theorem has been applied to study the alarm transmission delay induced by the uncoordinated sleeping schemes in wireless sensor networks [9].

Let T_i denote the random variable associated with the geographic link l_{ij} between SU i and SU j. T_i is the time during which the link l_{ij} is not functional so that the two SUs cannot connect to each other. According to Definition 1, the expectation of T_i is upper and lower bounded by

$$E(T_i) < 2\pi(r^2 + R^2)\lambda_p\Pi_1 E(\tau_1) = \overline{T}_{\max} \quad (16)$$

and

$$E(T_i) > E(\tau_1) = \overline{T}_{\min} \quad (17)$$

respectively. Consider any two SUs u and v located at X_u and X_v, respectively. Let

$$T(X_u, X_v) = \inf_{w(X_u,X_v)}\left\{\sum_{i\in V(w(X_u,X_v))} T_i\right\} \quad (18)$$

where $w(X_u, X_v)$ is an arbitrary path between u and v. $V(w(X_u, X_v))$ is the set of SUs along the path $w(X_u, X_v)$. Therefore, $T(X_u, X_v)$ is the smallest message delay on the path from u to v.

Theorem 2 *Consider $G(\lambda_s, r, S_p(t))$ with $\lambda_s^* > \lambda_s > \lambda_s^c$ and $\lambda_p > \lambda_p^c$. If two SUs, uandv, belong to the infinite connected component $G(\lambda_s, r)$, then there exists a strictly positive value $E(T_i) < \rho < \infty$, such that*

$$\Pr\left\{\lim_{||X_u - X_v||\to\infty} \frac{T(X_u, X_v)}{||X_u - X_v||} = \rho\right\} = 1 \quad (19)$$

Theorem 2 states that the latency in the secondary network $G(\lambda_s, r, S_p(t))$ is asymptotically linear in the Euclidean distance between the sender and the receiver if the sender and the receiver are in the infinite connected component of $G(\lambda_s, r)$. Note that the constant only depends on the network parameters including Π_1, r, R, λ_p, and $E(\tau_1)$, and is independent from the random locations of primary and secondary users.

It is also important to notice that ρ is lower bounded by $E(T_i)$, and the lower and upper bounds of $E(T_i), T_{\min}$ and $T_{\max}$, are increasing functions of the network parameters mentioned above. This means that the message transmission delay between two given locations could be increased under three conditions. (a) More PUs join the primary network such that λ_p increases. (b) PUs have more data to transmit, thus leading to larger active probability Π_1 and higher average transmission time $E(\tau_1)$. (c) PUs(SUs) adjust the transmission power to enlarge the transmission radius $R(r)$. Since the three conditions could decrease the available spectrum in the secondary network, larger transmission delay is expected and consistent with the statement of Theorem 2.

Now, we introduce some notations used through the rest of this section. We denote by C_∞ the infinite connected component in $G(\lambda_s, r)$. For each coordinate $(i, 0)$ with $i \in Z$, denote the location of the nearest SU in C_∞ by $\widetilde{X}_i$, i.e., $\widetilde{X}_i = \arg\min_{X_j \in C_\infty} ||X_j| - (i, 0)||$. Let $T_{m,n} = T(\widetilde{X}_m, \widetilde{X}_n)$.

To prove Theorem 2, we need the following proposition

Proposition 2

$$\lim_{n\to\infty}\left(\frac{T_{0,n}}{n}\right) = \rho \tag{20}$$

with probability one
where $\rho = \lim_{n\to\infty}\frac{E(T_{0,n})}{n} = \inf_{n\geq 1}\frac{E(T_{0,n})}{n}$

To show Proposition 2, we use Liggetts subadditive ergodic theorem [10] *as follows.*

Theorem 3 (subadditive ergodic theorem) *Let* $\{T_{m,n}\}$ *be a collection of random variables indexed by integers satisfying* $0 \leq m < n$. *Suppose* $\{T_{m,n}\}$ *has following properties: (i)* $T_{0,n} \leq T_{0,m} + T_{m,n}$. *(ii) The distribution of* $\{T_{m,m+k} : k \geq 1\}$ *does not depend onm. (iii)* $\{T_{nk,(n+1)k} : n \geq 0\}$ *is a stationary sequence for each* $k \geq 1$. *(iv)* $E(|T_{0,n}|) < \infty$ *for each n. Then, (a)* $\eta = \lim_{n\to\infty}\frac{E(T_{0,n})}{n} = \inf_{n\geq 1}\frac{E(T_{0,n})}{n}$. *(b)* $T = \lim_{n\to\infty}\frac{T_{0,n}}{n}$ *with probability one and* $E(T) = \eta$. *Furthermore, if (v) for* $k \geq 1$, $\{T_{nk,(n+1)k} : n \geq 0\}$ *are ergodic, then (c)* $T = \eta$.

If all conditions (i)–(v) of Theorem 3 are verified, then we directly obtain Proposition 2. It is easy to see that condition (i) is satisfied because $T_{0,n}$ is defined as the smallest transmission delay between the SUs located at $\widetilde{X}_m$ and $\widetilde{X}_n$, and it is easy to see $T_{0,n}$ cannot exceed $T_{0,m} + T_{m,n}$. Moreover, the conditions (ii) and (iii) are clearly fulfilled because the secondary network is driven from the homogeneous Poisson point process, which itself is stationary. In the following section, we show that condition (iv) and (v) are also satisfied by proving the next two Lemmas.

Lemma 4 *Suppose that two SUs,* $\widetilde{X}_0$ *and* $\widetilde{X}_n$, *have the mutual distance* $d_{0,n} = ||\widetilde{X}_0 - \widetilde{X}_n|| < \infty$, *and they belong to the same infinite connected component in* $G(\lambda_s, r)$. *Then,* $E(|T_{0,n}|) = E(|T(\widetilde{X}_0, \widetilde{X}_n)|) < \infty$.

Proof To compute the upper bound of $E(|T_{0,n}|)$, we consider the shortest path (in hops) $L_{0,n}$ from $\widetilde{X}_0$ to $\widetilde{X}_n$. Denote $|L_{0,n}|$ the number of hops on such a path, and T_i the delay on each hop i. Since the smallest delay $T_{0,n}$ cannot be greater than the delay on any particular path, $E(|T_{0,n}|)$ is upper bounded by

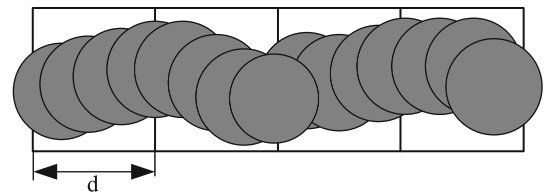

Fig. 4 Open rectangle

$$E(|T_{0,n}|) \leq E\left(\sum_{i=1}^{|L_{0,n}|} T_i\right) = E(T_i)E(|L_{0,n}|) \tag{21}$$

By (16), $E(T_i) < \infty$. Therefore, to show $E(|T_{0,n}|) < \infty$, is sufficient to prove $E(|L_{0,n}|) < \infty$. Towards this, $G(\lambda_s, r)$ is first mapped to a square lattice with the edge length $d_{0,n}$. Then, a sequence $\{G_i(d_{0,n})\}_{i\geq 1}$ of annuli around the origin is constructed in the same way as described in the Proof of Lemma 3. More specifically, for each annulus $G_i(d_{0,n})$, its four rectangles $A_i^+(d_{0,n})$, $A_i^-(d_{0,n})$, $B_i^+(d_{0,n})$, and $B_i^-(d_{0,n})$ can be determined according to Eq. (2), in which we subtitute $d_{0,n}/2$ for d_p. As a consequence, $\{G_i(d_{0,n})\}_{i\geq 1}$ have the similar contour as depicted in Fig. 2 except that the edge length in this case is $d_{0,n/2}$ instead of d_p.

We declare $A_i^+(d_{0,n})(A_i^-(d_{0,n}))$ to be open if $A_i^+(d_{0,n})$ $(A_i^-(d_{0,n}))$ is crossed from left to right by a connected component in $G(\lambda_s, r)$. We declare $B_i^+(d_{0,n})$, and $B_i^-(d_{0,n})$ to be open in the same way, except that the rectangle is crossed from bottom to top. A example of the open rectangle is given in Fig. 4. Note that the open rectangle defined here is different from the closed rectangle shown in Fig. 1 because they are crossed by the connected components of different networks, $G(\lambda_s, r)$ and $G(\lambda_p, 2R)$, respectively.

Let $\widetilde{A}_i^+(d_{0,n}), \widetilde{A}_i^-(d_{0,n}), \widetilde{B}_i^+(d_{0,n})$, and $\widetilde{B}_i^-(d_{0,n})$ be the event that $A_i^+(d_{0,n}), A_i^-(d_{0,n}), B_i^+(d_{0,n})$, and $B_i^-(d_{0,n})$ are open, respectively. Since $\lambda_s > \lambda_s^c$, by Corollary 4 in [11], we know for any $0 < \delta < 1$, there always exists $i < \infty$ so that $\Pr(\widetilde{A}_i^+(d_{0,n})) = \Pr(\widetilde{A}_i^-(d_{0,n})) = \Pr(\widetilde{B}_i^+(d_{0,n})) = \Pr(\widetilde{B}_i^-(d_{0,n})) = \Pr^i \geq \delta$. More specifically, we can find a value N such that if $i > N, \delta > (\frac{3}{4})^{1/4}$, i.e.,

$$N = \inf\left\{i : \overset{i}{\Pr} \geq \delta > \frac{3}{4}^{1/4}\right\} \tag{22}$$

Denote $\widetilde{G}_i(d_{0,n})$ the event that all the four rectangles of an annulus $G_i(d_{0,n})$ are open. Since $\widetilde{A}_i^+(d_{0,n}), \widetilde{A}_i^-(d_{0,n}), \widetilde{B}_i^+(d_{0,n})$ and $\widetilde{B}_i^-(d_{0,n})$ are increasing events, by the FKG inequality [11], we have $\Pr(\widetilde{G}_i(d_{0,n})) = \Pr(\widetilde{A}_i^{+}(d_{0,n})) \cap \Pr(\widetilde{A}_i^{-}(d_{0,n})) \cap \Pr(\widetilde{B}_i^{+}(d_{0,n})) \cap \Pr(\widetilde{B}_i^{-}(d_{0,n})) \geq \delta^4$.

Now, a key observation is that if all the four rectangles of an annulus $G_i(d_{0,n})$ are open, i.e., $\widetilde{G}_i(d_{0,n})$ occurs, then the annulus $G_i(d_{0,n})$ contains an open circuit. Consequentially, the shortest path $L_{0,n}$ is necessarily included in the square $[-d_{0,n}2^{i-1}, d_{0,n}2^{i-1}] \times [-d_{0,n}2^i, d_{0,n}2^i]$. (Otherwise, $L_{0,n}$ would go outside the square, and thus intersect with the open circuit in $G_i(d_{0,n})$ such that a path shorter than $L_{0,n}$ can be found. This, however, contradicts the claim that $L_{0,n}$ is the shortest.)

Based on the observation above, the number of hops on $L_{0,n}$, e.g., $|L_{0,n}|$, can be upper bounded by a certain value. To prove this, we need another fact that to generate the shortest path, any node on this path should connect to the farthest possible node. This means that if we put a ball of radius $r/2$ centered at each node on $L_{0,n}$, only the balls of consecutive nodes on $L_{0,n}$ can touch each other. Since $L_{0,n}$ is included in a square of the size of $4^i d_{0,n}$, $|L_{0,n}|$ is upper bounded by $|L_{0,n}| \leq 4^i d_{0,n}^2/(\pi(\frac{1}{2}r)^2)$. Therefore, $\Pr\left(|L_{0,n}| \leq \frac{4^{i+1}d_{0,n}^2}{\pi r^2}\right) \geq \Pr(\bigcup_{j=1}^{i} \widetilde{G}_j(d_{0,n}))$ This implies

$$\Pr\left(|L_{0,n}| > \frac{4^{i+1}d_{0,n}^2}{\pi r^2}\right) < \Pr\left(\bigcap_{j=1}^{i} \widetilde{G}_j(d_{0,n})\right) < (1-\delta^4)^i$$

Let $m = \frac{4^{N+1}d_{0,n}^2}{\pi r^2}$. $E(|L_{0,n}|)$ can be upper bounded by

$$\begin{aligned} E(|L_{0,n}|) &= \sum_{k=0}^{\infty} \Pr(|L_{0,n}| > k) \\ &= \sum_{k=0}^{m-1} \Pr(|L_{0,n}|) > k + \sum_{k=m}^{\infty} \Pr(|L_{0,n}|) > k \qquad (23) \\ &\leq m + \sum_{k=m}^{\infty} \frac{12 d_{0,n}^2 4^k}{\pi r^2} (1-\delta^4)^k < \infty \end{aligned}$$

The last inequality follows from the fact that the last summation is finite, as $\delta > (\frac{3}{4})^{1/4}$ by (22). By combining (21) and (23), we have $E(|T_{0,n}|) < \infty$. This completes the proof. □

The following lemma is to prove that condition (v) of Theorem 2 is satisfied, i.e., $\{T_{nk,(n+1)k} : n \geq 0\}$ is ergodic. As in [3], we prove $\{T_{nk,(n+1)k} : n \geq 0\}$ is fixing (asymptotically independent) instead, which is a stronger statement.

Lemma 5 The sequence $T_{nk,(n+1)k} : n \geq 0$ is mixing.

Proof Construct two squares S_1 and S_2 centered at $(nk, 0)$ and $((n+m)k, 0)$, respectively. Each square has an edge length of $(mk - 2\max(R, r))$. As m goes to infinity, both squares are included in the infinite connected component C_∞ with probability one. Moreover, by Lemma 4, we have $E(|T_{0,n}|) < \infty$ This implies $\Pr(T_{nk,(n+1)k} < \infty) = 1$ and $\Pr(T_{(n+m)k,(n+m+1)k} < \infty) = 1$. Therefore, we have

$$\lim_{m\to\infty} \Pr(T_{nk,(n+1)k} < t) = \Pr(T_{nk,(n+1)k} < t)$$

and

$$\begin{aligned} &\lim_{m\to\infty} \Pr(T_{(m+n)k,(n+m+1)k} < t') \\ &\quad = \Pr(T_{(n+m)k,(n+m+1)k} < t') \end{aligned}$$

On the other hand, as a consequence of Lemma 4, the path from $\widetilde{X}_0$ to $\widetilde{X}_n$ with the shortest delay $T_{0,n}$ is finite. Therefore, as m goes to infinity, the path with $T_{nk,(n+1)k}$ and the path with $T_{(n+m)k,(n+m+1)k}$ are included in the square S_1 and S_2, respectively. Moreover, according to the construction of S_1 and S_2, the right boundary of S_1 and the left boundary of S_2 are separated by a distance of $2\max(R, r)$. Thus, the SUs in S_1 and the SUs in S_2 do not have overlapped correlation regions (explained previously in Fig. 3). This implies the states of the SUs in S_1 are independent from the states of the SUs in S_2. As a consequence, $T_{nk,(n+1)k}$ and $T_{(n+m)k,(n+m+1)k}$ are independent of each other. The statement above is demonstrated in Fig. 5. Finally, we can show that the sequence $\{T_{nk,(n+1)k} : n \geq 0\}$ is mixing as follows:

$$\begin{aligned} &\lim_{m\to\infty} \Pr(T_{nk,(n+1)k} < t \cap T_{(n+m)k,(n+m+1)k} < t') \\ &\quad = \lim_{m\to\infty} \Pr(T_{nk,(n+1)k} < t) \cap \Pr(T_{(n+m)k,(n+m+1)k} < t') \\ &\quad = \Pr(T_{nk,(n+1)k} < t) \cap \Pr(T_{(n+m)k,(n+m+1)k} < t') \end{aligned}$$

This completes the proof. □

Proof (of Proposition 2) So far, we proved that $T_{m,n}$ satisfies all conditions of Theorem 3, and thus proved Proposition 2. More specifically, we showed that $T_{m,n}$ satisfies conditions (i) (ii) (iii) of Theorem 3 since $T_{m,n}$ is defined in a stationary manner. Lemma 4 proves $E(|T_{0,n}|) < \infty$, and thus guarantees that the condition (iv) also holds. The condition (v) is satisfied due to Lemma 5, which proves that $\{T_{nk,(n+1)k} : n \geq 0\}$ is mixing, and thus ergodic. □

Proof (of Theorem 2) Without loss of generality, take a straight line passing through X_u and X_v as the x-axis. Consider X_u as the origin. This means $X_u = \widetilde{X}_0$. We denote by n the integer closest to the x-axis coordinate of X_v. This means that $||X_v - (n, 0)|| < \frac{1}{2}$, and thus

$$||X_u - X_v|| = n + \Delta_d \qquad (24)$$

where $-\frac{1}{2} < \Delta_d < \frac{1}{2}$. Let d_n be the Euclidean distance between $\widetilde{X}_n$ and $(n, 0)$, i.e., $d_n = ||\widetilde{X}_n - (n, 0)||$. We have $d_n < \infty$ with probability one, because the probability that there is no node within a circle of range d centered at $(n, 0)$ is $e^{-\pi d^2 \lambda_s}$ and $e^{-\pi d^2 \lambda_s} \to 0$ as $d \to \infty$. Thus, by triangle

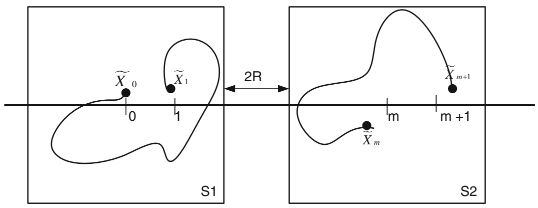

Fig. 5 The path yielding the smallest delay from $\widetilde{X}_0$ to $\widetilde{X}_1$ resides in square S_1, and the path yielding the smallest delay from $\widetilde{X}_m$ to $\widetilde{X}_{m+1}$ is in square S_2. As $m \to \infty$, the two paths are independent from each other

inequality, we have $||X_v - \widetilde{X}_n|| < ||\widetilde{X}_n - (n,0)|| + ||X_v - (n,0)|| < d_n + \frac{1}{2} < \infty$. Consequently, by Lemma 4, we obtain $T(X_v, \widetilde{X}_n) < \infty$. As a result, $T(X_u, X_v) > T_{0,n} - T(X_v, \widetilde{X}_n)$, and thus

$$T(X_u, X_v) = T_{0,n} - \Delta_t, \tag{25}$$

where $\Delta_t < T(X_v, \widetilde{X}_n) < \infty$. By (24) and (25), we obtain

$$\lim_{||X_u - X_v|| \to \infty} \frac{T(X_u, X_v)}{||X_u - X_v||} = \lim_{n \to \infty} \frac{T_{0,n}}{n}$$

Thus, from Proposition 2, we finally obtain

$$\Pr\left(\lim_{||X_u - X_v|| \to \infty} \frac{T(X_u, X_v)}{||X_u - X_v||} = \rho \right) = 1 \tag{26}$$

Now, we derive the upper and lower bounds of ρ. From Proposition 2, ρ is upper bounded by

$$\rho = \inf_{n \geq 1} \frac{E(T_{0,n})}{n} \leq E(T_{0,1}) < \infty \tag{27}$$

To prove the upper bound $\rho > 0$, let $d_0 = ||\widetilde{X}_0 - (n,0)||$ and $d_n = ||\widetilde{X}_n - (n,0)||$ It is clear that $d_0 < \infty$ and $d_n < \infty$ because $\Pr(d_0 = \infty) = \Pr(d_n = \infty) = e^{-\pi r^2 d_0} = 0$ as $d_0 \to \infty$. By triangle inequality, we have $||\widetilde{X}_n - \widetilde{X}_0|| > n - d_0 - d_n$. Thus, the number of hops from $\widetilde{X}_n$ to $\widetilde{X}_0$ is at least $\frac{n - d_0 - d_n}{r}$. Therefore, ρ is upper bounded by

$$\rho = \inf_{n \geq 1} \frac{E(T_{0,n})}{n} \geq \lim_{n \to \infty} \frac{E(T_i)(n - d_0 - d_n)}{rn} = \frac{E(T_i)}{r} > 0 \tag{28}$$

The last inequality holds because of (17). □

5 Conclusions

In this paper, the dynamic connectivity of large-scale cognitive radio networks are studied under the time-varying spectrum environment. It is shown that there exists a critical density λ_s^* such that if the density of λ_s of secondary network is larger than λ_s^*, the secondary network can maintain connectivity or percolation at all times even under the dynamically changing radio environment, i.e., there always exists an infinite connected component at each time $t > 0$ in the secondary network with probability one. In addition, the upper and lower bounds of λ_s^* are determined, and it is proven that they only depend on the network settings, such as primary and secondary network density, transmission radius, and active probability of primary users. Furthermore, it is shown that even when the secondary network is disconnected (in the subcritical phase) at all times, it is still possible for a SU to transfer its message to any destination with a certain delay with probability one. It is proven that this delay is asymptotically linear in the Euclidean distance between the transmitter and receiver.

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References

1. Akyildiz, I. F., Lee, W. Y., & Chowdhury, K. (2009). CRAHNs: Cognitive radio ad hoc networks. *Ad Hoc Networks (Elsevier) Journal, 7*(2), 810–836.
2. Cheng, Y.-C., & Robertazzi, T. G. (1989). Critical connectivity phenomena in multihop radio models. *IEEE Transaction on Communication, 37*, 770–777.
3. Dousse, O., Baccelli, F., & Thiran, P. (2004). Latency of wireless sensor networks with uncoordinated power saving mechanisms. In *In Proceedings of ACM MOBIHOC'04* (pp. 109–120).
4. Dousse, O., Baccelli, F., & Thiran, P. (2005). Impact of interferences on connectivity of ad hoc networks. *IEEE/ACM Transactions on Networking, 13*(2), 425–436.
5. Grimmett, G. (1999). *Percolation* (2nd ed.). New York: Springer.
6. Gupta, P., & Kumar, P. R. (1998). Critical power for asymptotic connectivity in wireless networks. *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W.H. Fleming, 7*(5), 547–566.
7. Haenggi, M., Andrews, J. G., Baccelli, F., Dousse, O., & Franceschetti, M. (2009). Stochastic geometry and random graphs for the analysis and design of wireless networks. *IEEE Journal on Selected Areas in Communications, 27*, 1029–1046.
8. Häggström, O., Peres, Y., & Steif, J. E. (1997). Dynamic percolation. *Annales de l'IHP Probabilités et Statistiques, 33*, 497–528.
9. Kong, Z., & Yeh, E. M. (2007). A distributed energy management algorithm for large-scale wireless sensor networks. In *Proceedings of ACM MOBIHOC'07* (pp. 209–218).
10. Liggett, T. (1985). An improved subadditive ergodic theorem. *Annals of Probability, 13*, 1279–1285.
11. Meester, R., & Roy, R. (1996). *Continuum percolation* (1st ed.). New York, NY, USA: Cambridge University Press.
12. Quintanilla, J., Torquato, S., & Ziff, R. M. (2000) Efficient measurement of the percoaltion threshold for fully penetrable discs. *Physics A, 86*, 399–407.
13. Ren, W., Zhao, Q., & Swami, A. (2009). Connectivity of cognitive radio networks: Proximity vs. opportunity. In *Proceedings of ACM MobiCom workshop on cognitive radio networks* (pp. 37–42).

14. Ren, W., Zhao, Q., & Swami, A. (2010). On the connectivity and multihop delay of ad hoc cognitive radio networks. In *Proceedings of ICC'2010* (pp. 1–6).
15. Santi, P., & Blough, D. M. (2002). An evaluation of connectivity in mobile wireless ad hoc networks. In *Proceedings of IEEE DSN* (pp. 89–102).
16. Wang, P., Akyildiz, I. F., & Al-Dhelaan, A. M. (2010). Dynamic connectivity of cognitive radio ad-hoc networks with time-varying spectral activity. In *Proceedings of Globecom 2010*.

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